Math 53: Multivariable Calculus **Sections 102, 103**

Handout for 2020-02-03

Conceptual questions

Question 1. If $\mathbf{u}$ and $\mathbf{v}$ are vectors of lengths 2 and 3 respectively, what are the largest and smallest possible values of $\mathbf{u} \cdot \mathbf{v}$? Draw pictures for both of these situations.

Question 2. If $\mathbf{r} = \langle x, y \rangle$, $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, what kind of shape does the equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ define in the xy-plane?

Question 3. With notation as in the preceding question, what kind of shape does the equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} = 0$ define in the xy-plane?

Computations

Problem 1. Let A be the point with coordinates $(1, 3, -2)$ and let B be the point with coordinates $(4, 6, -2)$.

(a) What is the vector $\overrightarrow{AB}$?

(b) Let C be the point $(3, 7, 2)$, and let D be the point on the line $\overline{AC}$ which is closest to B (so that $\overline{BD}$ and $\overline{AC}$ meet at a right angle). Find the vector $\overrightarrow{AD}$.

(c) Find the vector $\overrightarrow{BD}$.

(d) Find the vector $\overrightarrow{AE}$, which is obtained by "reflecting" $\overrightarrow{AB}$ across the line $\overline{AC}$.

See Figure 1.

Problem 2. There are no equilateral triangles in the plane $\mathbb{R}^2$ whose three vertices all have rational number coordinates. (For example, if $(0, 0)$ and $(2, 0)$ were two of the three vertices, then the third would be $(\sqrt{3}, \pm 1)$, and $\sqrt{3}$ is irrational.) I'm sure you agree that having to deal with irrational numbers is somewhat inconvenient for vector computations.

Luckily, there are actually equilateral triangles in space $\mathbb{R}^3$ whose vertices all have integer coordinates!

(a) Verify that the triangle with vertices $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$ is equilateral (all sides the same length).

(b) In particular, that means all three internal angles are $\pi/3$ (60 degrees). Verify this fact using the dot product.

One can also construct a regular (i.e. completely symmetric, all sides the same length) tetrahedron in $\mathbb{R}^3$ whose vertices all have integer coordinates.

(a) Take D to be the point $(1, 1, 1, \cancel{1})$. Verify that the distance from D to each of the points A, B, C is the same as the side length you computed for the equilateral triangle $\triangle ABC$ earlier.

(b) Find the coordinates of the center of the tetrahedron $ABCD$. (Take the average of the corners.)

(c) Call the center O. Use the dot product to compute the angle $\angle AOB$. (If you've taken chemistry, you may have learned that a tetrahedral molecule has a bond angle of this value; this problem shows you how to derive that fact.)

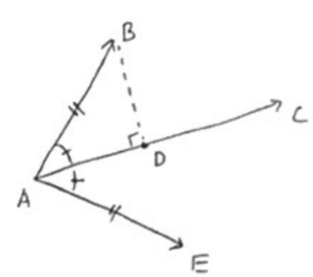
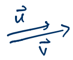


FIGURE 1

Questions:

1) Largest possible is 6: 

Smallest possible is -6: 

2) Circle with diameter connecting the points (a_1, a_2), (b_1, b_2)

3) Line through (a_1, a_2) perpendicular to the vector $\langle b_1, b_2 \rangle$

Computations:

1) a) $\langle 3,3,0 \rangle$ b) $\overrightarrow{AD} = \text{proj}_{\overrightarrow{AC}} \overrightarrow{AB} = \ldots$

c) $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = \ldots$

d) $\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{BD}$ or $\overrightarrow{AB} + 2\overrightarrow{BD}$.

2) a) Each side has length $\sqrt{2}$.

b) For example: $\overrightarrow{AB} = \langle -1,1,0 \rangle$, $\overrightarrow{AC} = \langle -1,0,1 \rangle$

angle: $\cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \dfrac{1}{2}$ so $\theta = \dfrac{\pi}{3}$.

a) omitted b) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

c) $\overrightarrow{OA} = \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle$ $\overrightarrow{OB} = \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle$

$\cos\theta = \dfrac{-1/4}{3/4} = -\dfrac{1}{3}$. $\theta \approx 109.5^\circ$